Benha University Faculty of Engineering- Shoubra Eng. Mathematics & Physics Department **Qualifying Courses (Mathematics)**		Final Term Exam Date: 14 – 1 – 2015 Course: Linear Algebra EMM 402 Duration: **3** hours

• Answer **All** questions The exam consists of one page • No. of questions: 5 Total Mark: 200

Question	Marks
[1] (a)Determine the linearly independent and linearly dependent: (i) u = (2, 2), v = (1, 3) (ii) u = (2, 1, 2), v = (1, 2, 0), w = (3, 3, 2)	10
(b) If $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & 3 \\ 1 & 3 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & -1 \\ 1 & 5 & 1 \end{bmatrix}$ Find, if possible, A + B, A + B^t, A.B, A.B^t, A.C, |A| and |C|.	40
[2](a)If $A = \begin{bmatrix} 2 & 6 \\ 2 & 3 \end{bmatrix}$. (i)Find the eigenvalues and the eigenvectors.	20
(ii)Find the eigenvalues of the matrices $B = f(A) = 2^A$ and $C = f(A) = A^4$	10
(b) Show that the eigenvalues of : $A = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$ are real numbers, where a, b, c are real numbers.	10
[3]Write the following expressions in matrix form and determine the type: (a) $P = 3x^2 + 4y^2 + 2z^2 + 2xy - 2xz + yz$ (b) $P = 2xy + 4xz - 2yz - 3x^2 - 2y^2 - 2z^2$ (c) $P = 2xy + 6xz - 2yz + 3x^2 + y^2 + z^2$	30
[4](a)Write the Hessian matrix of : $f(x, y, z) = ye^{2x} + \cos^3 y + x^3 \sin z^5$.	10
(b) Show that the eigenvectors of the matrix $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ are orthogonal and linear independent.	20
(c) Find $f(A) = \frac{24}{A+I}$ where $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$	20
[5](a)Write the equations: $a_{11}x + a_{12}y + a_{13}z = b_1$, $a_{21}x + a_{22}y + a_{23}z = b_2$, $a_{31}x + a_{32}y + a_{33}z = b_3$ in matrix form and discuss the types of solutions. Also, state two methods with their procedures for solving this linear system.	20
(b)Determine the type of solution of the linear system: 2x – y + 3z = 2, x + 3y – 4z = 3, 3x + 2y – z = 5	10

Examiners Board : Dr. Mohamed Eid Dr. Fathi Abdsallam Dr. Zaki Ahmed

Model Answer

Question 1

(a)(i) If a, b are constants.
Then a.u + b.v = a(2, 2) + b(1, 3) = (2a + b, 2a + 3b) = 0
Then 2a + b = 0 , 2a + 3 b = 0. The solution of these equations is a = b = 0
Then u, v are linearly independent.

(ii) If a, b , c are constants.
Then a.u + b.v + c.w = a(2, 1, 2) + b(1, 2, 0) + c(3, 3, 2)

$$= (2a + b + 3c,\ \ a + 2b + 3c,\ \ 2a + 0b + 2c) = 0$$

Then $2a + b + 3c = 0,\ \ a + 2b + 3c = 0,\ \ 2a + 0b + 2c = 0$
The solution of these equations is c = – a, b = a
Then u, v , w are linearly dependent.
--*10 Marks*

(b) If $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & 3 \\ 1 & 3 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & -1 \\ 1 & 5 & 1 \end{bmatrix}$

$A + B = \begin{bmatrix} 3 & 2 & 4 \\ 1 & 7 & 5 \end{bmatrix}$

$A + B^t$ does not exist , A.B does not exist

$A.B^t = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 9 \\ 9 & 18 \end{bmatrix}$

$A.C = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & -1 \\ 1 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 13 & 1 \\ 3 & 27 & -1 \end{bmatrix}$

|A| does not exist
|C| = (3 + 5) – 2(0 + 1) + 2(0 – 3) = 0
--*40 Marks*

Question 2

(a)(i) $|A - \lambda I| = \begin{vmatrix} 2-\lambda & 6 \\ 2 & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda) - 12 = \lambda^2 - 5\lambda - 6 = 0$

Then, the eigenvalues are: $\lambda_1 = 6,\ \ \lambda_2 = -1.$

From the equation, $\begin{bmatrix} 2-\lambda & 6 \\ 2 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$

For : $\lambda_1 = 6$, $\begin{bmatrix} -4 & 6 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$

Then – 4x + 6y = 0, 2x – 3y = 0

For : $\lambda_2 = -1$, $\begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$

Then 3x + 6y = 0, 2x + 4y = 0

Then $2x = 3y =$ any number except 0

Put $y = 2$, we get $x = 3$ and the corresponding eigenvector is: $X_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Then $x = -2y =$ any number except 0

Put $y = 1$, we get $x = -2$ and the corresponding eigenvector is: $X_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

-- ***(20 Marks)***

(ii)The eigenvalues of the matrix $B = f(A) = 2^A$ are $2^6,\ 2^{-1}$

The eigenvalues of the matrix $C = f(A) = A^4$ are $6^4,\ (-1)^4$

-- ***(10 Marks)***

(b) Since $|A - \lambda I| = \begin{vmatrix} a-\lambda & c \\ c & b-\lambda \end{vmatrix} = (a-\lambda)(b-\lambda) - c^2 = 0$

Then, the characteristic equation is $\lambda^2 - (a+b)\lambda + ab - c^2 = 0$

The root of this equation are :

$$\lambda = \frac{(a+b) \pm \sqrt{(a+b)^2 - 4(ab - c^2)}}{2} = \frac{1}{2}[(a+b) \pm \sqrt{a^2 + b^2 - 2ab + 4c^2}]$$

$$= \frac{1}{2}[(a+b) \pm \sqrt{(a-b)^2 + 4c^2}]$$

Since $[(a-b)^2 + 4c^2] > 0$

Then the eigenvalues are real numbers

-- ***(10 Marks)***

<u>Question 3</u>

(a) $P = X`AX = [x \quad y \quad z]\begin{bmatrix} 3 & 1 & -1 \\ 1 & 4 & \frac{1}{2} \\ -1 & \frac{1}{2} & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

We see that all principal minors are positive numbers. Then P is positive definite.

-- ***(10 Marks)***

(b) $P = X`AX = [x \quad y \quad z]\begin{bmatrix} -3 & 1 & 2 \\ 1 & -2 & -1 \\ 2 & -1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

We see that all principal minors of second order are positive numbers.

We see that all principal minors of first and third order are negative numbers.

Then P is negative definite.

-- ***(10 Marks)***

(c) $P = X`AX = [x \quad y \quad z]\begin{bmatrix} 3 & 1 & 3 \\ 1 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

We see that all principal minors of second order are not positive numbers.
Then P is indefinite.

-- ***(10 Marks)***

Question 4

(a)The Hessian matrix is :

$$H = \begin{bmatrix} 4ye^{2x} + 6x\sin z^5 & 2e^{2x} & 15x^2z^4\cos z^5 \\ 2e^{2x} & 6\cos y\,(\sin y)^2 - 3(\cos y)^3 & 0 \\ 15x^2z^4\cos z^5 & 0 & 20x^3z^3\cos z^5 - 25x^3z^8\sin z^5 \end{bmatrix}$$

-- ***(10 Marks)***

(b) $|A - \lambda I| = \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = (2-\lambda)(2-\lambda) - 1 = \lambda^2 - 4\lambda + 3 = 0$

Then, the eigenvalues are: $\lambda_1 = 1, \quad \lambda_2 = 3.$

From the equation, $\begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$

For : $\lambda_1 = 1$, $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$

Then x – y = 0, – x + y = 0

Then x = y = any number except 0

Put y = 1, we get x = 1 and the

corresponding eigenvector is: $X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

For : $\lambda_2 = 3$, $\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$

Then – x – y = 0, – x – y = 0

Then x = – y = any number except 0

Put y = 1, we get x = – 1 and the

corresponding eigenvector is: $X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

We see that $X_1`.X_2 = [1 \quad 1]\begin{bmatrix} -1 \\ 1 \end{bmatrix} = -1 + 1 = 0$

Then, the eigenvectors are orthogonal and linear independent.

-- ***(20 Marks)***

(c) $|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 \\ 3 & 4-\lambda \end{vmatrix} = (2-\lambda)(4-\lambda) - 3 = \lambda^2 - 6\lambda + 5 = 0$

Then, the eigenvalues are: $\lambda_1 = 1, \quad \lambda_2 = 5.$

Since $f(\lambda_1) = 12, \; f(\lambda_2) = 4$

Then $P(\lambda) = f(\lambda_1)\frac{\lambda-5}{1-5} + f(\lambda_2)\frac{\lambda-1}{5-1} = 12\frac{\lambda-5}{1-5} + 4\,\frac{\lambda-1}{5-1} = 14 - 2\lambda$

Then $f(A) = \frac{24}{A+I} = 14I - 2A = \begin{bmatrix} 14 & 0 \\ 0 & 14 \end{bmatrix} - 2\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 10 & -2 \\ -6 & 6 \end{bmatrix}$

--***20 Marks***

Question 5

(a)The equations written as :

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

The types of solutions are : one solution, infinite number of solutions, no solution.

The methods of solutions this linear system are : Gauss method, Crammer method.

--***20 Marks***

(b) The augmented matrix of the linear system is:

$$G = \left[\begin{array}{ccc:c} 1 & 3 & -4 & 3 \\ 2 & -1 & 3 & 2 \\ 3 & 2 & -1 & 5 \end{array}\right].$$

Apply the operations, $-2R_1 + R_2$, $-3R_1 + R_3$ and then R2 – R3

$$G \sim \left[\begin{array}{ccc:c} 1 & 3 & -4 & 3 \\ 0 & -7 & 11 & -4 \\ 0 & -7 & 11 & -4 \end{array}\right] \sim \left[\begin{array}{ccc:c} 1 & 3 & -4 & 3 \\ 0 & -7 & 11 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We see that, rank A = 2 = rank G . Then, it has infinite number of solutions.

--***10 Marks***

Dr. Mohamed Eid